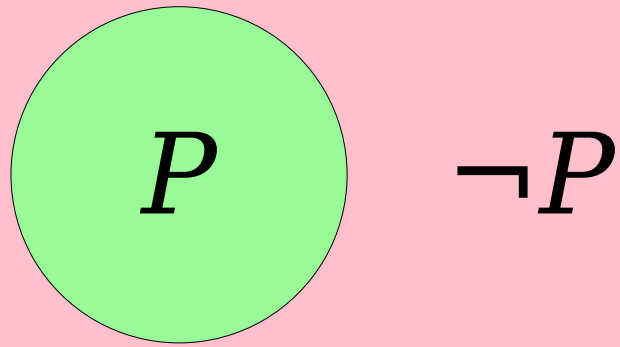
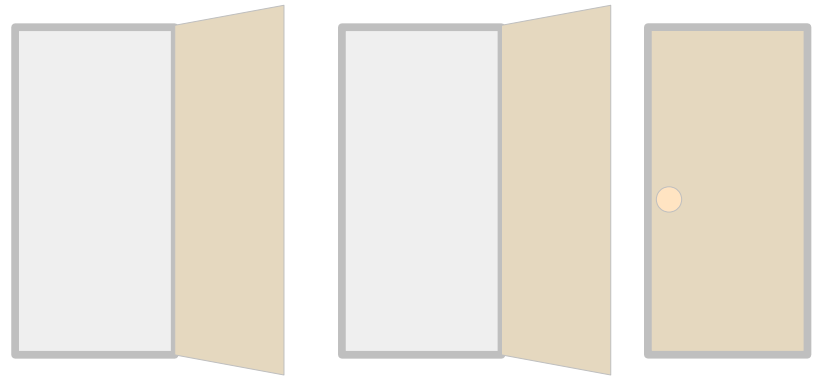


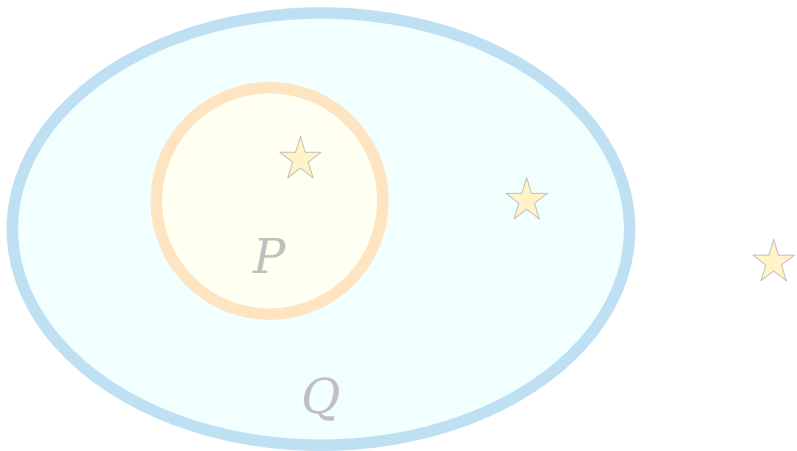
Indirect Proofs



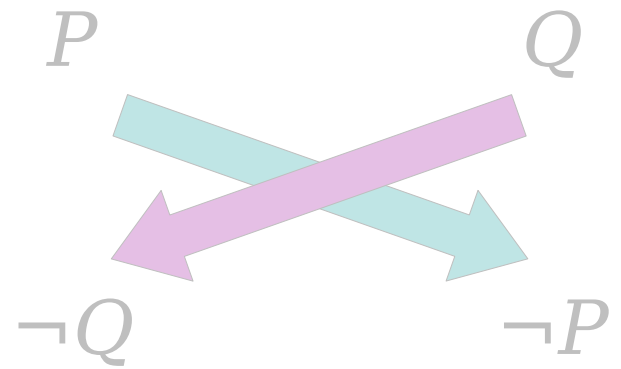
Logical Negation



Proof by Contradiction



Logical Implication



Proof by Contrapositive

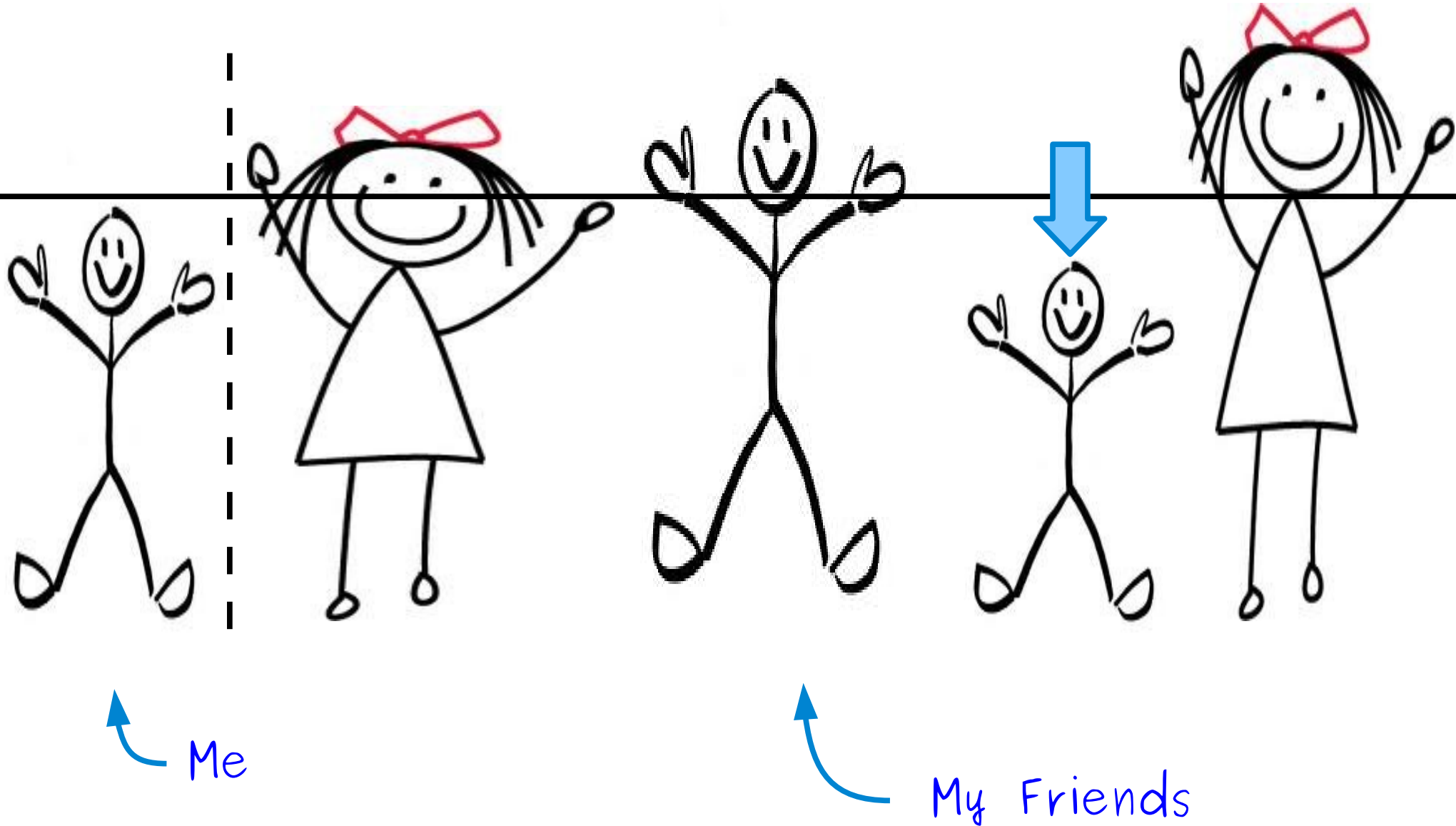
Logical Negation

Negations

- A **proposition** is a statement that is either true or false.
- Some examples:
 - If n is an even integer, then n^2 is an even integer.
 - $\emptyset = \mathbb{R}$.
- The **negation** of a proposition X is a proposition that is true when X is false and is false when X is true.
- For example, consider the proposition “it is snowing outside.”
 - Its negation is “it is not snowing outside.”
 - Its negation is *not* “it is sunny outside.”
 - Its negation is *not* “we’re in the Bay Area.”

How do you find the negation
of a statement?

“All My Friends Are Taller Than Me”



The negation of the *universal* statement

Every P is a Q

is the *existential* statement

There is a P that is not a Q .

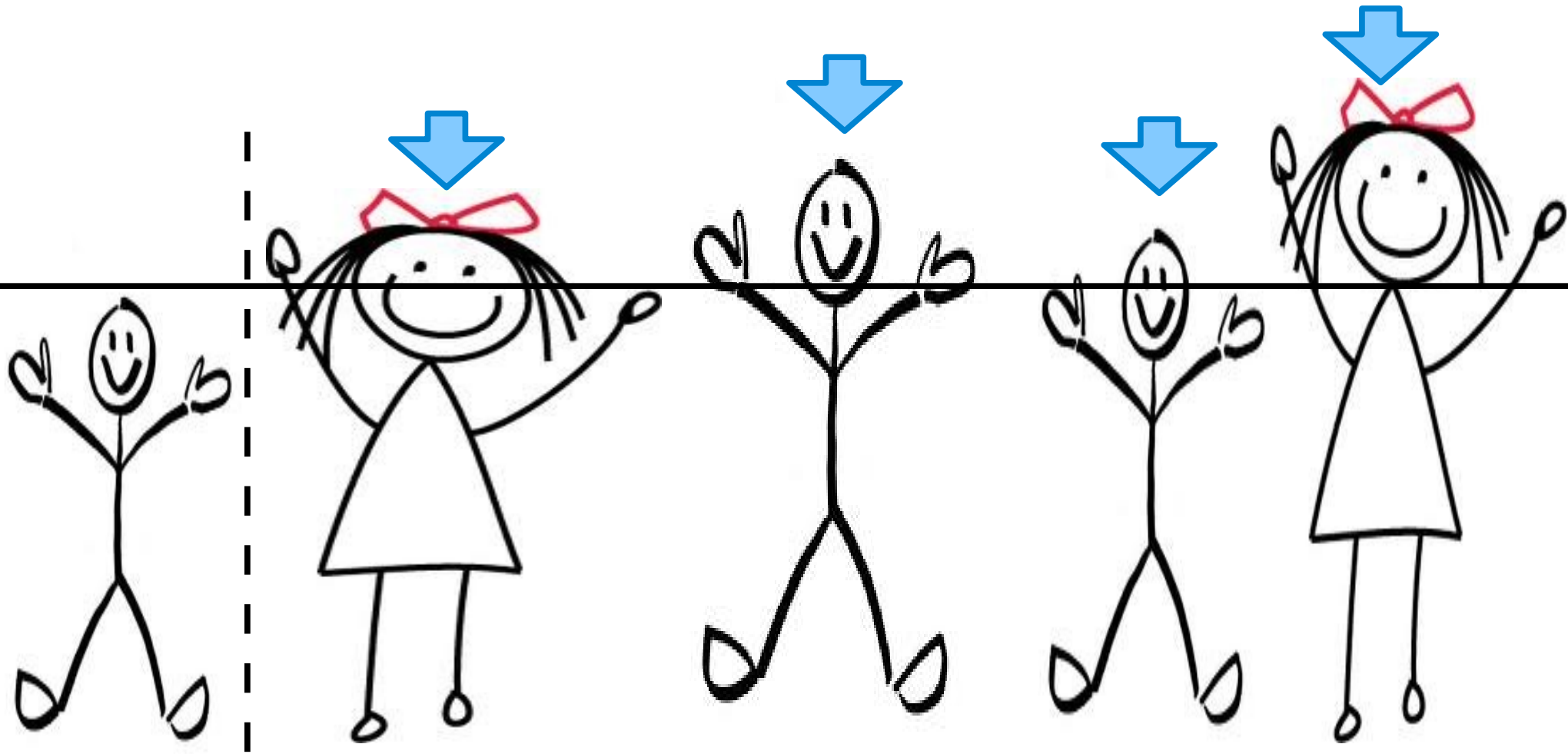
The negation of the *universal* statement

For all x , $P(x)$ is true.

is the *existential* statement

There exists an x where $P(x)$ is false.

“Some Friend Is Shorter Than Me”



Me

My Friends

The negation of the *existential* statement

There exists a P that is a Q

is the *universal* statement

Every P is not a Q .

The negation of the *existential* statement

There exists an x where $P(x)$ is true

is the *universal* statement

For all x , $P(x)$ is false.

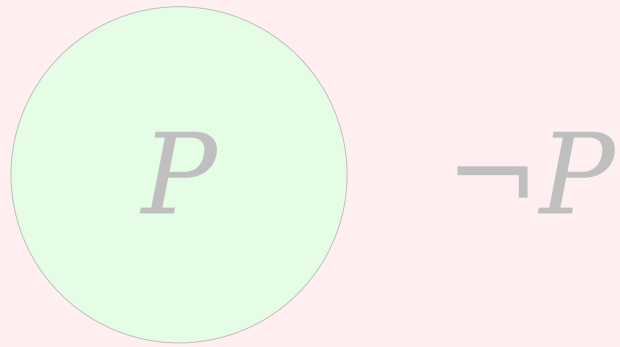
Your Turn!

- What's the negation of the following statement?

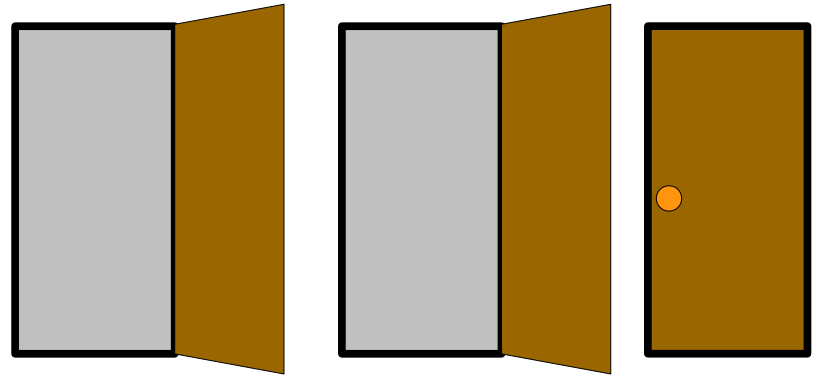
*“Every brown dog
loves every orange cat.”*

- Answer:

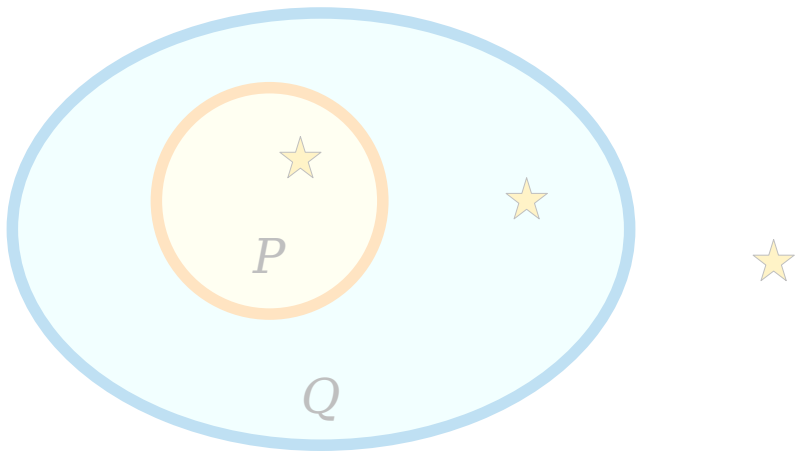
*“There is a brown dog
that doesn't love
some orange cat”*



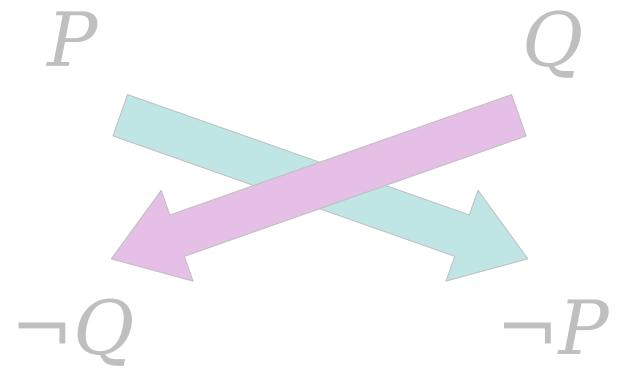
Logical Negation



Proof by Contradiction



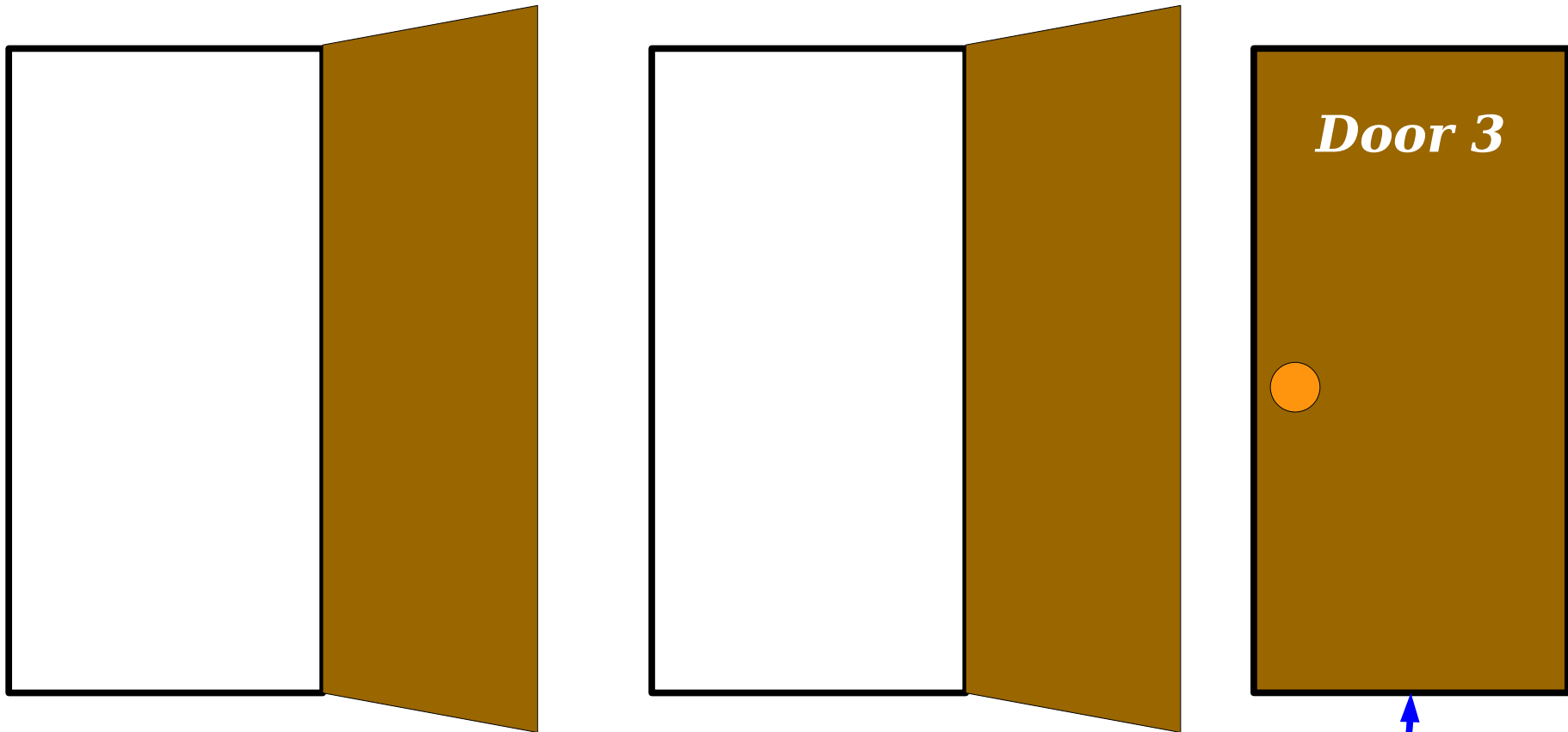
Logical Implication



Proof by Contrapositive

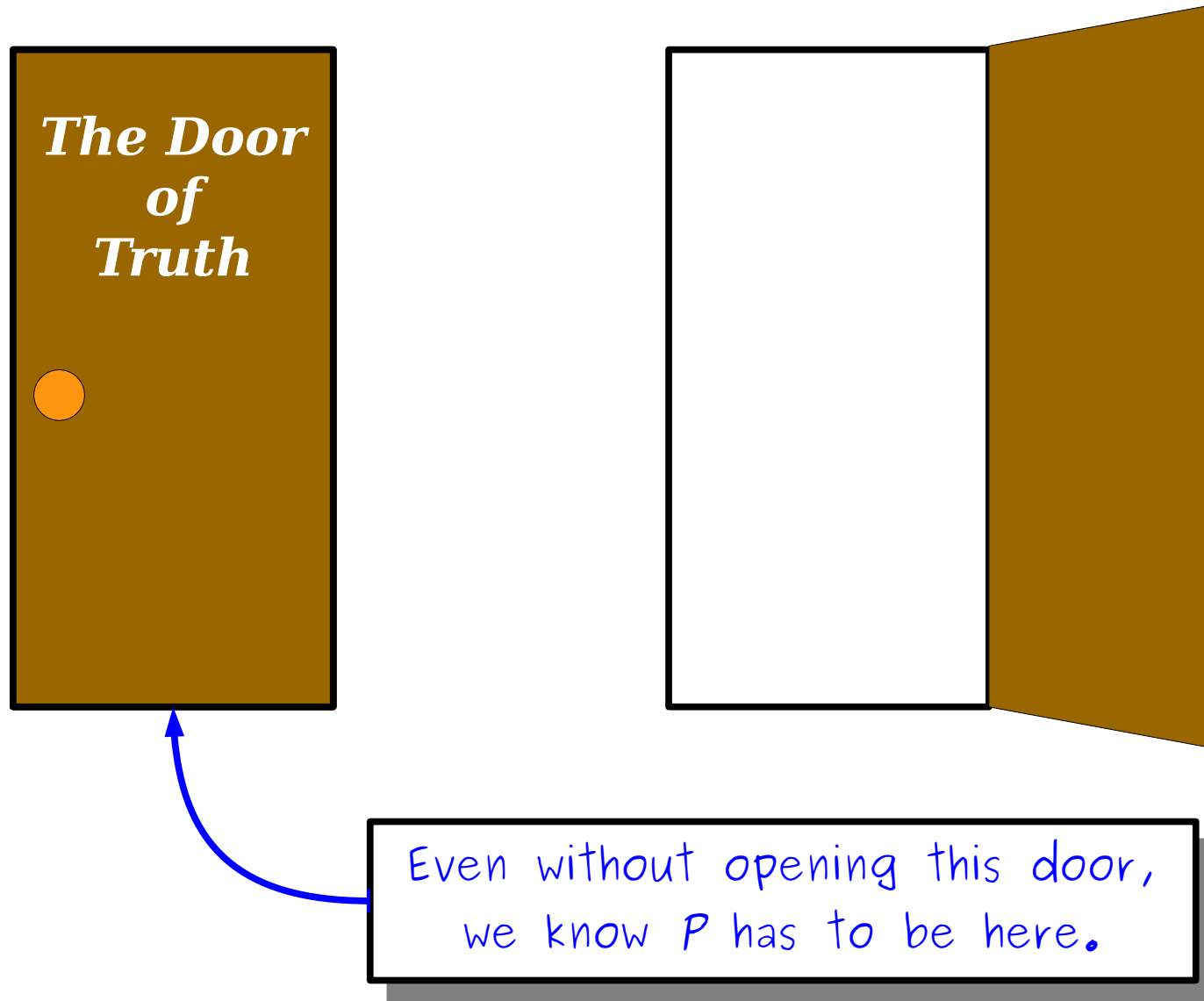
Proof by Contradiction

*There's something hidden behind one of these doors.
Which door is it hidden behind?*



Even without opening this door, we know whatever is hidden has to be here.

*Every statement in mathematics is either true or false.
If statement P is not false, what does that tell you?*



A ***proof by contradiction*** shows that some statement P is true by showing that P isn't false.

Proof by Contradiction

- **Key Idea:** Prove a statement P is true by showing that it isn't false.
- First, assume that P is false. The goal is to show that this assumption is silly.
- Next, show this leads to an impossible result.
 - For example, we might have that $1 = 0$, that $x \in S$ and $x \notin S$, that a number is both even and odd, etc.
- Finally, conclude that since P can't be false, we know that P must be true.

An Example: ***Set Cardinalities***

Set Cardinalities

- We've seen sets of many different cardinalities:
 - $|\emptyset| = 0$
 - $|\{1, 2, 3\}| = 3$
 - $|\{n \in \mathbb{N} \mid n < 137\}| = 137$
 - $|\mathbb{N}| = \aleph_0$.
 - $|\wp(\mathbb{N})| > |\mathbb{N}|$
- These span from the finite up through the infinite.
- **Question:** Is there a “largest” set? That is, is there a set that's bigger than every other set?

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S .

To prove this statement by contradiction, we're going to assume its negation.

What is the negation of the statement "there is no largest set?"

One option: "there is a largest set."

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S .

Notice that we're announcing

1. that this is a proof by contradiction, and
2. what, specifically, we're assuming.

This helps the reader understand where we're going. Remember - proofs are meant to be read by other people!

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S .

Now, consider the set $\wp(S)$. By Cantor's Theorem, we know that $|S| < |\wp(S)|$, so $\wp(S)$ is a larger set than S . This contradicts the fact that S is the largest set.

We've reached a contradiction, so our assumption must have been wrong. Therefore, there is no largest set. ■

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S .

The three key pieces:

1. Say that the proof is by contradiction.
2. Say what you are assuming is the negation of the statement to prove.
3. Say you have reached a contradiction and what the contradiction means.

In CS103, please include all these steps in your proofs!

We've reached a contradiction, so our assumption must have been wrong. Therefore, there is no largest set. ■

Another Example

Latin Squares

- A **Latin square** is an $n \times n$ grid filled with the numbers $1, 2, \dots, n$ such that every number appears in each row and each column exactly once.
- The **main diagonal** of a Latin square runs from the top-left corner to the bottom-right corner.
- A Latin square is **symmetric** if the numbers are symmetric across the main diagonal.

1	2	3
2	3	1
3	1	2

1	2	3	4	5
2	5	4	1	3
3	4	2	5	1
4	1	5	3	2
5	3	1	2	4

3	2	5	1	4
2	1	4	5	3
5	4	2	3	1
1	5	3	4	2
4	3	1	2	5

2	5	1	4	3
5	1	3	2	4
1	3	4	5	2
4	2	5	3	1
3	4	2	1	5

Latin Squares

- Notice anything about what's on the main diagonals of these symmetric Latin squares?
- **Theorem:** Every odd-sized symmetric Latin square has every number $1, 2, \dots, n$ on its main diagonal.

1	2	3
2	3	1
3	1	2

1	2	3	4	5
2	5	4	1	3
3	4	2	5	1
4	1	5	3	2
5	3	1	2	4

3	2	5	1	4
2	1	4	5	3
5	4	2	3	1
1	5	3	4	2
4	3	1	2	5

2	5	1	4	3
5	1	3	2	4
1	3	4	5	2
4	2	5	3	1
3	4	2	1	5

Theorem: Every symmetric Latin square of odd size $n \times n$ has each of the numbers $1, 2, \dots, n$ on its main diagonal.

Proof:

What is the negation of the theorem?

Every symmetric Latin square of odd size $n \times n$ has each of the numbers $1, 2, \dots, n$ on its main diagonal.

One option:

There is a symmetric Latin square of odd size $n \times n$ that does not have one of the numbers $1, 2, \dots, n$ on its main diagonal.

Theorem: Every symmetric Latin square of odd size $n \times n$ has each of the numbers $1, 2, \dots, n$ on its main diagonal.

Proof: Assume for the sake of contradiction that there is a symmetric Latin square of odd size $n \times n$ that does not have one of the numbers $1, 2, 3, \dots, n$ on its main diagonal.

Notice that we're announcing

1. that this is a proof by contradiction, and
2. what, specifically, we're assuming.

This helps the reader understand where we're going. Remember - proofs are meant to be read by other people!

Theorem: Every symmetric Latin square of odd size $n \times n$ has each of the numbers $1, 2, \dots, n$ on its main diagonal.

Proof: Assume for the sake of contradiction that there is a symmetric Latin square of odd size $n \times n$ that does not have one of the numbers $1, 2, 3, \dots, n$ on its main diagonal. Call the missing number r .

Let k be the number of times r appears above the main diagonal. Since the Latin square is symmetric, there are also k copies of r below the main diagonal. And because r doesn't appear on the main diagonal, that accounts for all copies of r , so there are exactly $2k$ copies of r .

Independently, we know that r appears n times in the Latin square, once for each of its n rows.

Combining these results, we see that $n = 2k$. This means that n is even, contradicting the fact that n is odd. We've reached a contradiction, so our assumption was wrong. Therefore, all symmetric Latin squares of odd size $n \times n$ have each of the numbers $1, 2, \dots, n$ on their main diagonals. ■

Theorem: Every symmetric Latin square of odd size $n \times n$ has each of the numbers $1, 2, \dots, n$ on its main diagonal.

Proof: Assume for the sake of contradiction that there is a symmetric Latin square of odd size $n \times n$ that does not have one of the numbers $1, 2, 3, \dots, n$ on its main diagonal. Call the missing number r .

The three key pieces:

1. Say that the proof is by contradiction.
2. Say what you are assuming is the negation of the statement to prove.
3. Say you have reached a contradiction and what the contradiction means.

In CS103, please include all these steps in your proofs!

Independently, we know that r appears n times in the Latin square, once for each of its n rows.

Combining these results, we see that $n = 2k$. This means that n is even, contradicting the fact that n is odd. We've reached a contradiction, so our assumption was wrong. Therefore, all symmetric Latin squares of odd size $n \times n$ have each of the numbers $1, 2, \dots, n$ on their main diagonals. ■

Time-Out for Announcements!

Outdoor Activities

- You're less than fifty miles from grassy mountains, redwood forests, Pacific coastline, beautiful wetlands, and more.
- Want to explore the area to see what it has to offer? Check out our (unofficial) Outdoor Activities Guide.

https://cs103.stanford.edu/outdoor_activities

- A sampler of what to check out:
 - Drive to the observatory in the mountains near San Jose and take in the views.
 - Visit a beach with an enormous colony of elephant seals.
 - Walk in redwood forests and pick your own bay leaves.
 - Grab cheap, high-quality food from unassuming strip malls.

Vaccines!

- It's Vaccine Season! Yay! What a great way to protect yourself and others.
- You can get a free flu shot through Vaden. Details are at this link:

<https://ehs.stanford.edu/flu/information>

- Stanford Health Care offers free bivalent COVID booster vaccines. Use this link to create an account to sign up:

<https://myhealth.stanfordhealthcare.org/>

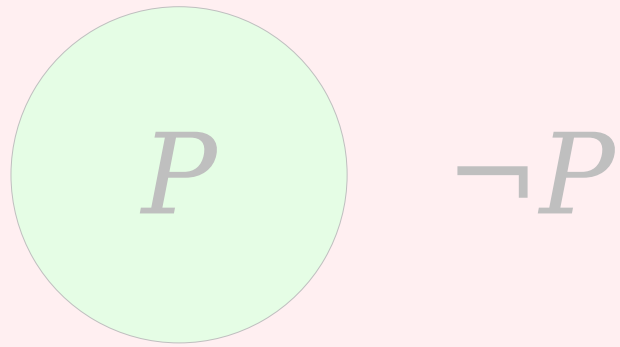
- Santa Clara County (where Stanford is located) also offers free flu shots, COVID vaccines, and COVID boosters. Details and appointments here:

<https://vax.sccgov.org/>

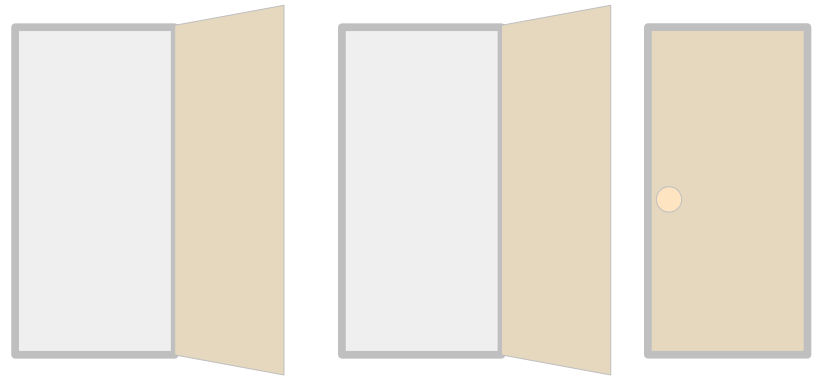
Readings for Today

- On the course website we have some information you should look over.
- First is the ***Proofwriting Checklist***. It contains information about style expectations for proofs. We'll be using this when grading, so be sure to read it over.
- Next is the ***Guide to Office Hours***, which talks about how our office hours work and how to make the most effective use of them.
- Finally is the ***Guide to LaTeX***, which explains how to use LaTeX to typeset your problem sets in a way that's so beautiful it will bring tears to your eyes.

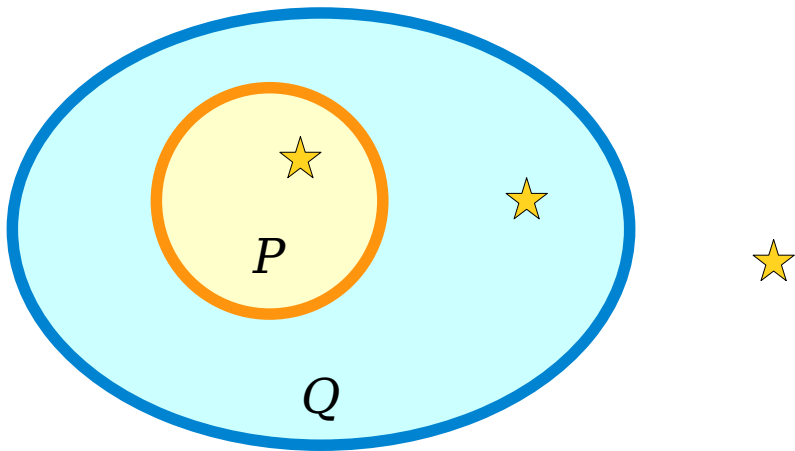
Back to CS103!



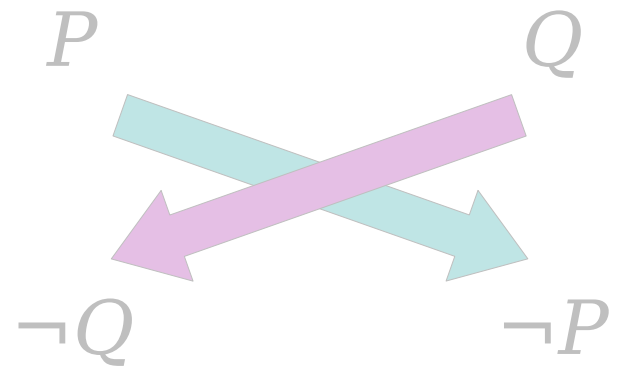
Logical Negation



Proof by Contradiction



Logical Implication



Proof by Contrapositive

If n is an even integer, then n^2 is an even integer.

This part of the implication is called the *antecedent*.

This part of the implication is called the *consequent*.

An ***implication*** is a statement of the form
“If P is true, then Q is true.”

If n is an even integer, then n^2 is an even integer.

If m and n are odd integers, then $m+n$ is even.

If you like the way you look that much,
then you should go and love yourself.

An ***implication*** is a statement of the form
“If P is true, then Q is true.”

What Implications Mean

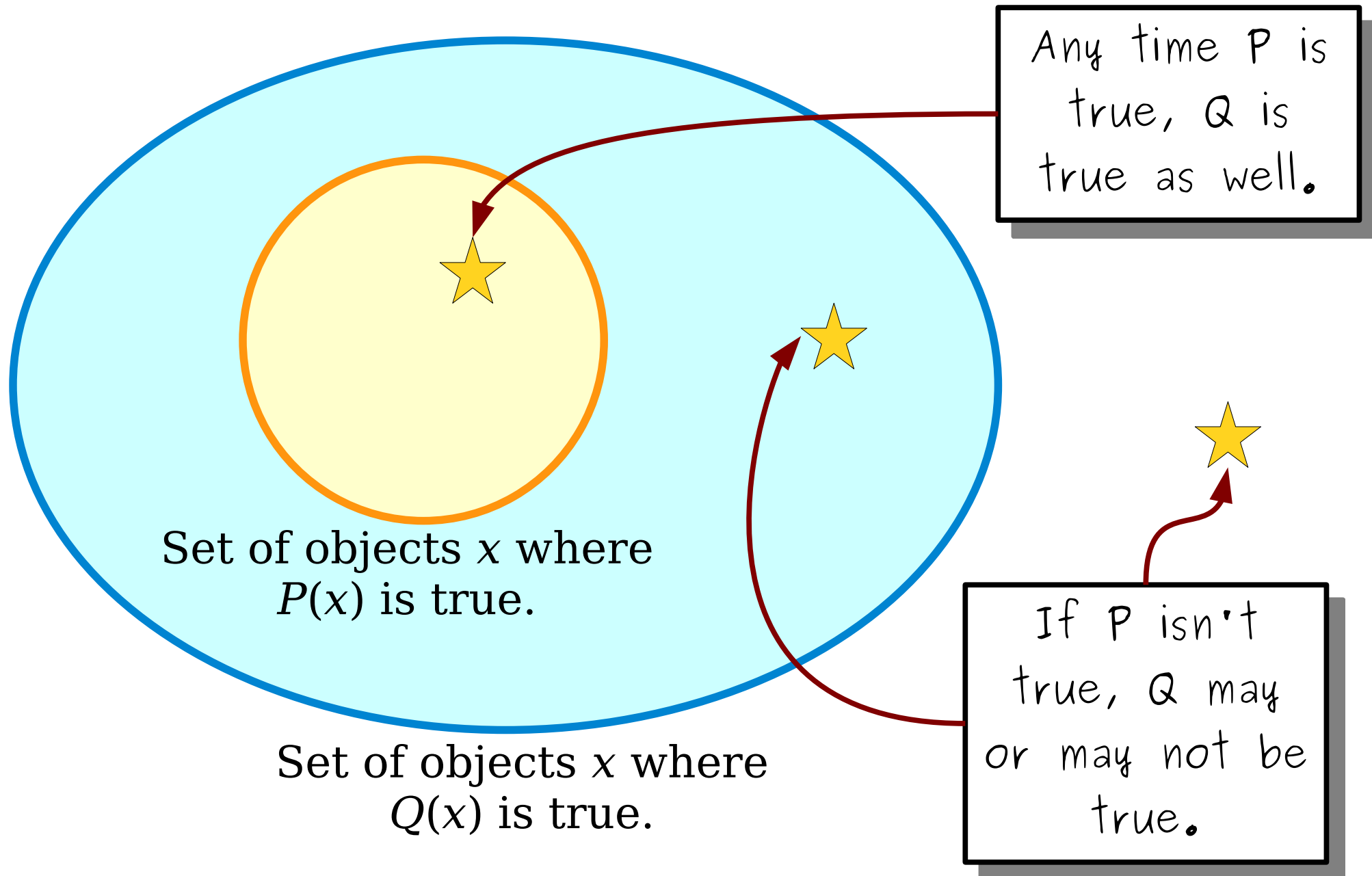
**“If there's a rainbow in the sky,
then it's raining somewhere.”**

- In mathematics, implication is directional.
 - The above statement doesn't mean that if it's raining somewhere, there has to be a rainbow.
- In mathematics, implications only say something about the consequent when the antecedent is true.
 - If there's no rainbow, it doesn't mean there's no rain.
- In mathematics, implication says nothing about causality.
 - Rainbows do not cause rain.

What Implications Mean

- In mathematics, a statement of the form **For any x , if $P(x)$ is true, then $Q(x)$ is true** means that any time you find an object x where $P(x)$ is true, you will see that $Q(x)$ is also true (for that same x).
- There is no discussion of causation here. It simply means that if you find that $P(x)$ is true, you'll find that $Q(x)$ is also true.

Implication, Diagrammatically



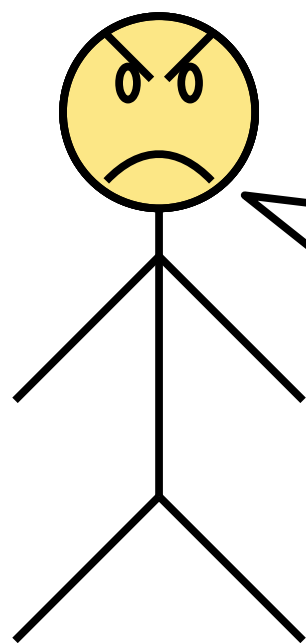
How do you negate an implication?



Story Time!

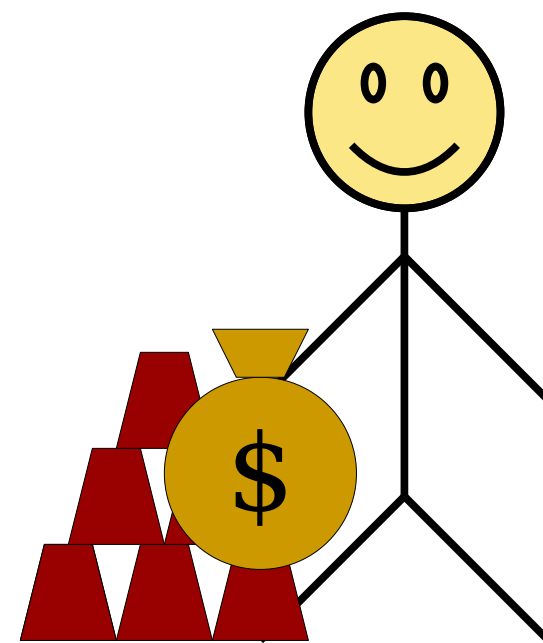
Ancient Contract:

If Nanni pays money to Ea-Nasir, then Ea-Nasir will give Nanni quality copper ingots.



Nanni

I'm going to complain about this!
(That's a hyperlink. Click it.)



Ea-Nasir

Question: What has to happen for this contract to be broken?

Answer: Nanni pays Ea-Nasir and doesn't get quality copper ingots.

The negation of the statement

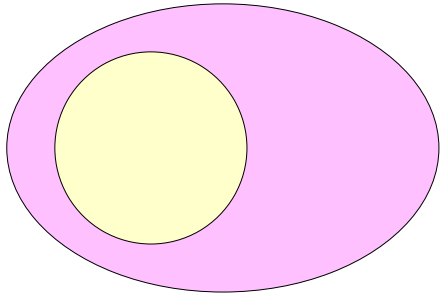
**“For any x , if $P(x)$ is true,
then $Q(x)$ is true”**

is the statement

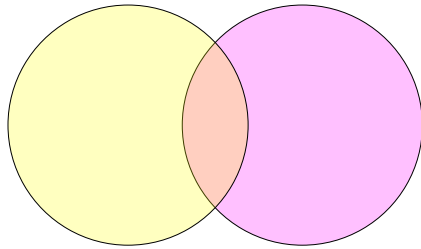
**“There is at least one x where
 $P(x)$ is true and $Q(x)$ is false.”**

***The negation of an implication
is not an implication!***

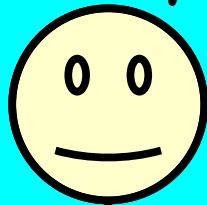
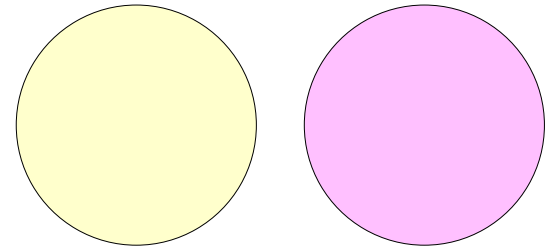
If p is a puppy,
then I do love p !



It's
complicated.



If p is a puppy,
then I don't love p !



How to Negate Universal Statements:

“For all x , $P(x)$ is true”

becomes

“There is an x where $P(x)$ is false.”

How to Negate Existential Statements:

“There exists an x where $P(x)$ is true”

becomes

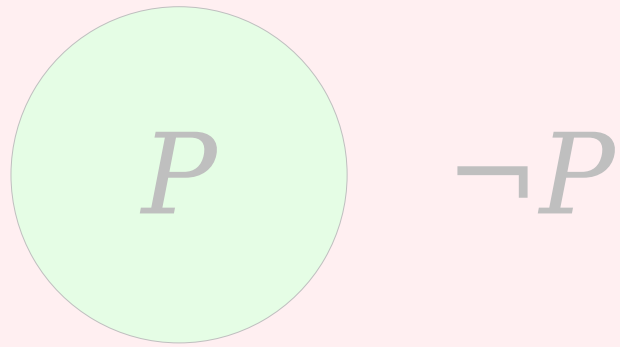
“For all x , $P(x)$ is false.”

How to Negate Implications:

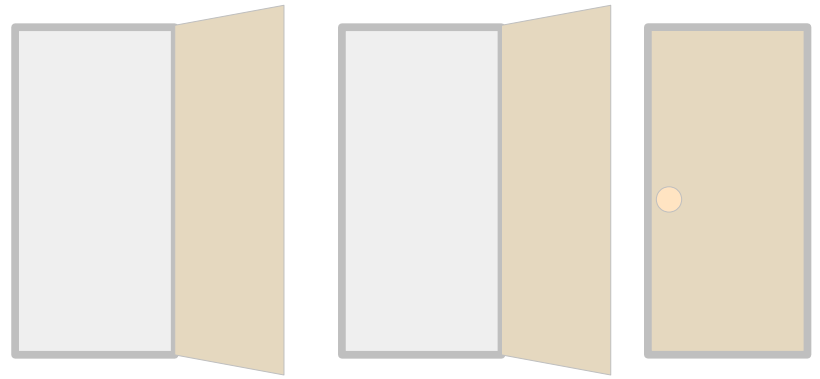
“For every x , if $P(x)$ is true, then $Q(x)$ is true”

becomes

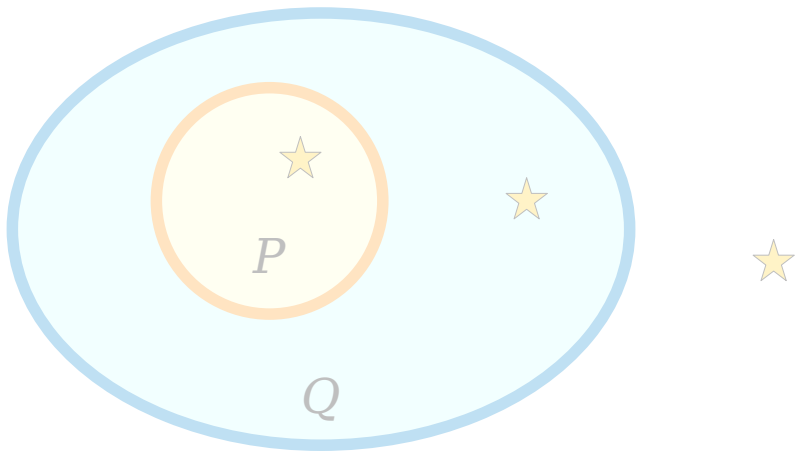
“There is an x where $P(x)$ is true and $Q(x)$ is false.”



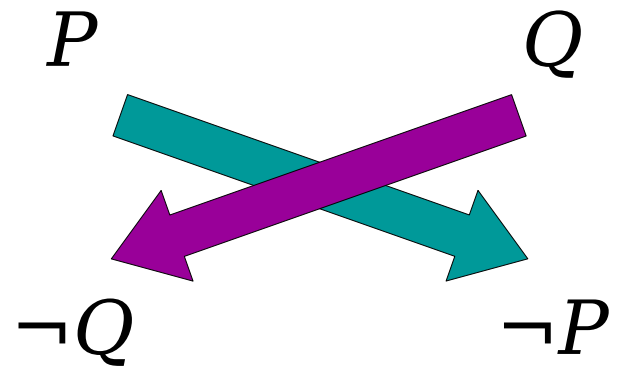
Logical Negation



Proof by Contradiction



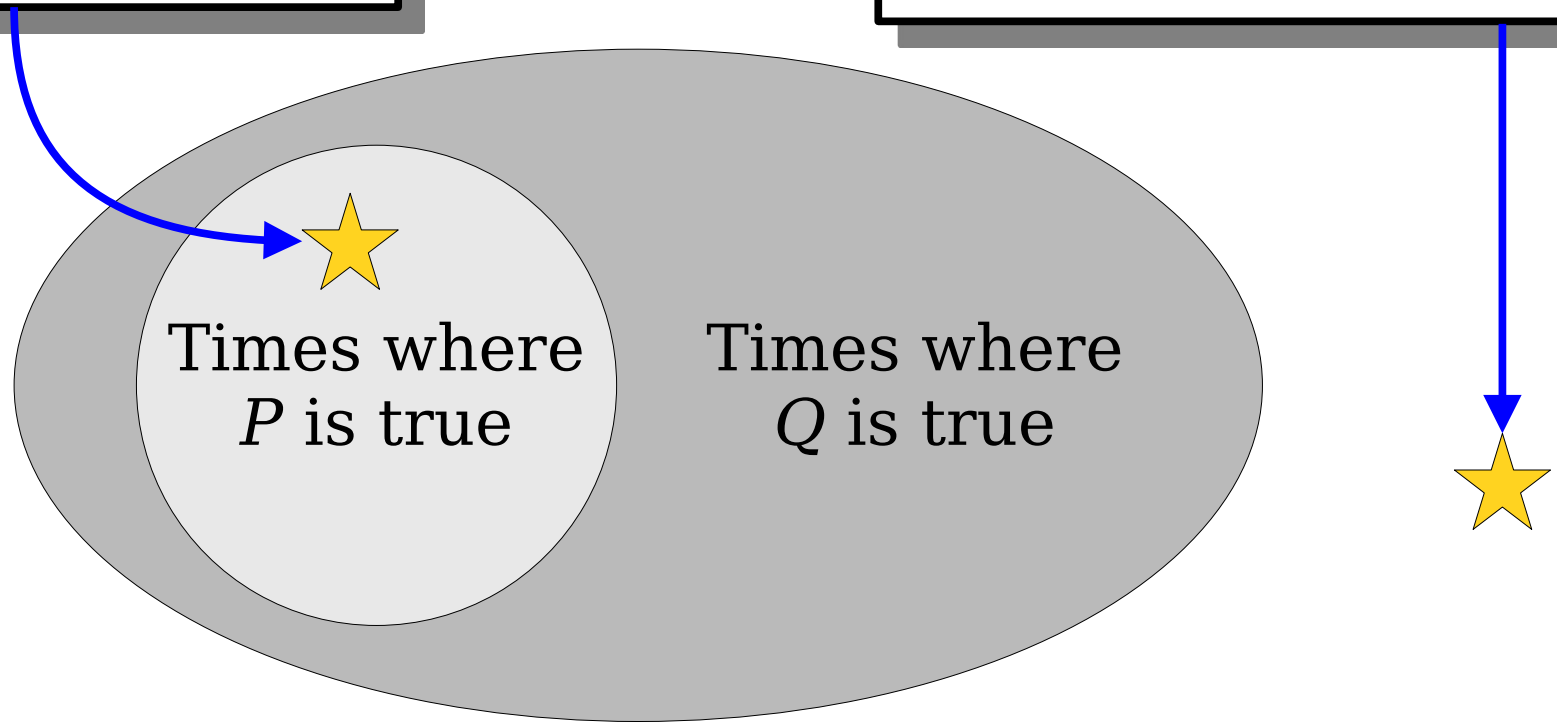
Logical Implication



Proof by Contrapositive

Anything inside this inner bubble is also inside the outer bubble.

Anything outside this outer bubble is outside the inner bubble.



If P is true, then Q is true.

If Q is false, then P is false.

The Contrapositive

- The **contrapositive** of the implication

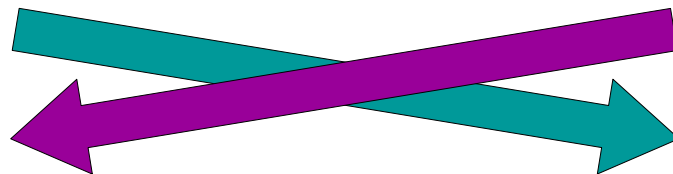
If **P is true**, then **Q is true**

is the implication

If **Q is false**, then **P is false**.

- The contrapositive of an implication means exactly the same thing as the implication itself.

If it's a puppy, then I love it.



If I don't love it, then it's not a puppy.

The Contrapositive

- The **contrapositive** of the implication

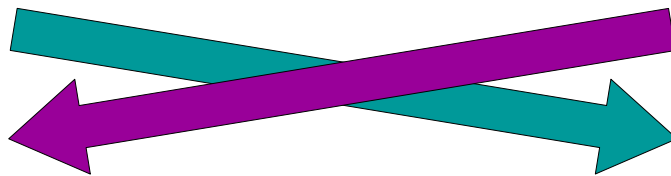
If **P is true**, then **Q is true**

is the implication

If **Q is false**, then **P is false**.

- The contrapositive of an implication means exactly the same thing as the implication itself.

If I store cat food inside, then raccoons won't steal it.



If raccoons stole the cat food, then I didn't store it inside.

To prove the statement

“if P is true, then Q is true,”

you can choose to instead prove the
equivalent statement

“if Q is false, then P is false,”

if that seems easier.

This is called a ***proof by contrapositive***.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement

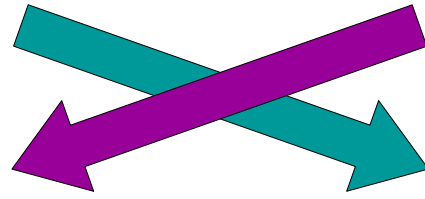
This is a courtesy to the reader and says "heads up! we're not going to do a regular old-fashioned direct proof here."

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd.

What is the contrapositive of this statement?

if n^2 is even, then n is even.



If n is odd, then n^2 is odd.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd.

Here, we're explicitly writing out the contrapositive. This tells the reader what we're going to prove. It also acts as a sanity check by forcing us to write out what we think the contrapositive is.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that **if n is odd, then n^2 is odd.**

We've said that we're going to prove this new implication, so let's go do it! The rest of this proof will look a lot like a standard direct proof.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd as well.

We know that n is odd, which means there is an integer k such that $n = 2k + 1$. This in turn tells us that

$$\begin{aligned}n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1.\end{aligned}$$

From this, we see that there is an integer m (namely, $2k^2 + 2k$) such that $n^2 = 2m + 1$. That means that n^2 is odd, which is what we needed to show. ■

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd.

We know
integer
us that

The general pattern here is the following:

1. Start by announcing that we're going to use a proof by contrapositive so that the reader knows what to expect.
2. Explicitly state the contrapositive of what we want to prove.
3. Go prove the contrapositive.

From th
(namely
means t
to show. ■

Biconditionals

- The previous theorem, combined with what we saw on Wednesday, tells us the following:

For any integer n , if n is even, then n^2 is even.

For any integer n , if n^2 is even, then n is even.

- These are two different implications, each going the other way.
- We use the phrase ***if and only if*** to indicate that two statements imply one another.
- For example, we might combine the two above statements to say
for any integer n : n is even if and only if n^2 is even.

Proving Biconditionals

- To prove a theorem of the form
 P if and only if Q ,
you need to prove two separate statements.
 - First, that if P is true, then Q is true.
 - Second, that if Q is true, then P is true.
- You can use any proof techniques you'd like to show each of these statements.
 - In our case, we used a direct proof for one and a proof by contrapositive for the other.

What We Learned

- ***How do you negate formulas?***
 - It depends on the formula. There are nice rules for how to negate universal and existential statements and implications.
- ***What's a proof by contradiction?***
 - It's a proof of a statement P that works by showing that P cannot be false.
- ***What's an implication?***
 - It's statement of the form “if P , then Q ,” and states that if P is true, then Q is true.
- ***What is a proof by contrapositive?***
 - It's a proof of an implication that instead proves its contrapositive.
 - (The contrapositive of “if P , then Q ” is “if not Q , then not P .”)

Your Action Items

- ***Read “Guide to Office Hours,” the “Proofwriting Checklist,” and the “Guide to LaTeX.”***
 - There’s a lot of useful information there. In particular, be sure to read the Proofwriting Checklist, as we’ll be working through this checklist when grading your proofs!
- ***Start working on PS1.***
 - At a bare minimum, read over it to see what’s being asked. That’ll give you time to turn things over in your mind this weekend.

Next Time

- ***Mathematical Logic***
 - How do we formalize the reasoning from our proofs?
- ***Propositional Logic***
 - Reasoning about simple statements.
- ***Propositional Equivalences***
 - Simplifying complex statements.

Appendix: Proving Implications by
Contradiction

Proving Implications

- Suppose we want to prove this implication:
 If P is true, then Q is true.
- We have three options available to us:
 - ***Direct Proof:***
 Assume **P is true**, then prove **Q is true**.
 - ***Proof by Contrapositive.***
 Assume **Q is false**, then prove that **P is false**.
 - ***Proof by Contradiction.***
 ... what does this look like?

Theorem: For any integer n , if n^2 is even, then n is even.

What is the negation of our theorem?

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Proof: Assume for the sake of contradiction that there is an integer n where n^2 is even, but n is odd.

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Proof: Assume for the sake of contradiction that there is an integer n where n^2 is even, but n is odd.

Since n is odd we know that there is an integer k such that

$$n = 2k + 1. \quad (1)$$

Squaring both sides of equation (1) and simplifying gives the following:

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned} \quad (2)$$

Equation (2) tells us that n^2 is odd, which is impossible; by assumption, n^2 is even.

We have reached a contradiction, so our assumption must have been incorrect. Thus if n is an integer and n^2 is even, n is even as well. ■

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The three key pieces:

1. Say that the proof is by contradiction.
2. Say what the negation of the original statement is.
3. Say you have reached a contradiction and what the contradiction entails.

In CS103, please include all these steps in your proofs!

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Proving Implications

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If **P is true**, then **Q is true**.

- We have three options available to us:

- ***Direct Proof:***

Assume **P is true**, then prove **Q is true**.

- ***Proof by Contrapositive.***

Assume **Q is false**, then prove that **P is false**.

- ***Proof by Contradiction.***

Assume **P is true** and **Q is false**,
then derive a contradiction.